

Sequential projective measurements for channel decoding

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We study the transmission of classical information in quantum channels. We present a decoding procedure that is very simple but still achieves the channel capacity. It is used to give an alternative straightforward proof that the classical capacity is given by the regularized Holevo bound. This procedure uses only projective measurements and is based on successive “yes”/“no” tests only.

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According to quantum information theory, to transfer classical signals we must encode them into the states of quantum information carriers, transmit these through the (possibly noisy) communication channel, and then decode the information at the channel output [1]. Frequently, even if no entanglement between successive information carriers is employed in the encoding or is generated by the channel, a joint measurement procedure is necessary (e.g. see [2]) to achieve the capacity of the communication line, i.e. the maximum transmission rate per channel use [1]. This is clear from the original proofs [3, 4] that the classical channel capacity is provided by the regularization of the Holevo bound [5]: these proofs employ a decoding procedure based on detection schemes (the ‘Pretty-Good-Measurement’ or its variants [6–17]). Alternative decoding schemes were also derived in [18] (with a combinatorial approach) and in [19–21] (with an application of quantum hypothesis testing, which was introduced in this context in [22]). Here we present a simple decoding procedure which uses only dichotomic projective measurements, but which is nonetheless able to achieve the channel capacity.

The main idea is that even if the possible alphabet states (i.e. the states of a single information carrier) are not orthogonal at the output of the channel, the codewords composed of a long sequence of alphabet states approach orthogonality asymptotically, as the number of letters in each codeword goes to infinity. Thus, one can sequentially test whether each codeword is at the output of the channel. When one gets the answer “yes”, the probability of error is small (as the other codewords have little overlap with the tested one). When one gets the answer “no”, the state has been ruined very little and can be still employed to further test for the other codewords. To reduce the accumulation of errors during a long sequence of tests that yield “no” answers, every time a “no” is obtained, we have to project the state back to the space that contains the typical output of the channel. Summarizing, the procedure is: 1. test whether the channel output is the first codeword; 2. if “yes”, we are done, if “no”, then project the system into the typical subspace and abort with an error if the projection fails; 3. Repeat

the above procedure for all the other codewords until we get a “yes” (or abort with an error if we test all of them without getting “yes”). 4. In the end, we identified the codeword that was sent or we had to abort.

We start reviewing some basic notions on typicality. Then, we prove that the above procedure achieves the classical capacity of the channel. An alternative (more formal) proof that refers to this same method is presented in [23] by using a decoding strategy in which the “yes/no” measurements discriminates only among the typical subspaces of the codewords. An application of our scheme to communication over Gaussian bosonic channels [24] will be presented in Ref. [25].

Definitions and review:— For notational simplicity we will consider codewords composed of unentangled states. For general channels, entangled codewords must be used to achieve capacity [26], but the extension of our theory to this case is straightforward (replacing the Holevo bound with its regularized version).

Consider a quantum channel that is fed with a letter j from a classical alphabet with probability p_j . The letter j is encoded into a state of the information carriers which is evolved by the channel into an output $\rho_j = \sum_k p_{k|j} |k\rangle_j \langle k|$, where ${}_j \langle k'|k\rangle_j = \delta_{k'k}$. Hence, the average output is

$$\rho = \sum_j p_j \rho_j = \sum_{j,k} p_j p_{k|j} |k\rangle_j \langle k| = \sum_k p_k |k\rangle \langle k|, \quad (1)$$

where $|k\rangle_j$ and $|k\rangle$ are the eigenvectors of the j th output-alphabet density matrix and of the average output respectively. The subtleties of quantum channel decoding arise because the ρ_j typically commute neither with each other nor with ρ . The Holevo-Shumacher-Westmoreland (HSW) theorem [3, 4] implies that we can send classical information reliably down the channel at a rate (bits per channel use) given by the Holevo quantity [5]

$$\chi \equiv S(\rho) - \sum_j p_j S(\rho_j), \quad (2)$$

where $S(\cdot) \equiv -\text{Tr}[(\cdot) \log_2(\cdot)]$ is the von Neumann entropy. This rate can be asymptotically attained in the multi-channel uses scenario as $\lim_{n \rightarrow \infty} (\log_2 N_n)/n$,

where a set \mathcal{C}_n of N_n codewords $\vec{j} = (j_1, \dots, j_n)$ formed by long sequences of the letters j are used to reliably transfer N_n distinct classical messages. Similarly to the Shannon random-coding theory [27], the codewords $\vec{j} \in \mathcal{C}_n$ can be chosen at random among the *typical sequences* generated by the probability p_j , in which each letter j of the alphabet occurs approximately $p_j n$ times. As mentioned in the introduction, the HSW theorem uses the ‘Pretty-Good-Measurement’ procedure to decode the codewords of \mathcal{C}_n at the output of the channel. We will now show that a sequence of binary projective measurements suffices [28].

Sequential measurements for channel decoding:— The channel output state $\rho_{\vec{j}} \equiv \rho_{j_1} \otimes \dots \otimes \rho_{j_n}$ associated to a generic typical sequence $\vec{j} = (j_1, \dots, j_n)$ possesses a *typical subspace* $\mathcal{H}_{\vec{j}}$ spanned by the vectors $|k_1\rangle_{j_1} \dots |k_n\rangle_{j_n} \equiv |\vec{k}\rangle_{\vec{j}}$, where $|k\rangle_j$ occurs approximately $p_j p_{k|j} n = p_{jk} n$ times, e.g. see Ref. [3]. The subspace $\mathcal{H}_{\vec{j}}$ has dimensions $\sim 2^{n \sum_j p_j S(\rho_j)}$ independent of the input $\vec{j} \in \mathcal{C}_n$. Moreover, a typical output subspace \mathcal{H} and a projector P onto it exist such that, for any $\epsilon > 0$ and sufficiently large n

$$\text{Tr } \bar{\rho} > 1 - \epsilon, \quad (3)$$

where $\bar{\rho} \equiv P \rho \otimes \dots \otimes \rho P$ is the projection of the n -output average density matrix onto \mathcal{H} . Notice that \mathcal{H} and the $\mathcal{H}_{\vec{j}}$ ’s in general differ. Typicality for \mathcal{H} implies that, for $\delta > 0$ and sufficiently large n , the eigenvalues λ_i of $\bar{\rho}$ and the dimension of \mathcal{H} are bounded as [3, 4]

$$\lambda_i \leq 2^{-n(S(\rho) - \delta)}, \quad (4)$$

$$\# \text{ nonzero eigenvalues} = \dim(\mathcal{H}) \leq 2^{n(S(\rho) + \delta)}. \quad (5)$$

Define then the operator

$$\tilde{\rho} = P \left(\sum_{\vec{j}, \vec{k} \in \text{typ}} p_{\vec{j}} p_{\vec{k}|\vec{j}} |\vec{k}\rangle_{\vec{j}} \langle \vec{k}| \right) P \leq \bar{\rho}, \quad (6)$$

where the inequality follows because the summation is only restricted to the \vec{j} ’s that are typical sequences of the classical source, and to the states $|\vec{k}\rangle_{\vec{j}}$ which span the typical subspace of the \vec{j} -th output. [Without these limitations, the inequality would be replaced by an equality.] Consequently the maximum eigenvalue of $\tilde{\rho}$ is no greater than that of $\bar{\rho}$ while the number of nonzero eigenvalues of $\tilde{\rho}$ cannot be greater than those of $\bar{\rho}$, i.e. Eqs. (3)–(5) apply also to $\tilde{\rho}$.

Now we come to our main result. To distinguish between the N_n distinct codewords of \mathcal{C}_n , we perform sequential von Neumann measurements corresponding to projections onto the possible outputs $|\vec{k}\rangle_{\vec{j}}$ to find the channel input (as shown in [23] these can also be replaced by joint projectors on the spaces $\mathcal{H}_{\vec{j}}$). In between these measurements, we perform von Neumann measurements that project onto the typical output subspace \mathcal{H} .

We will show that as long as the rate at which we send information down the channel is bounded above by the Holevo quantity (2), these measurements identify the proper input to the channel with probability one in the limit that the number of uses of the channel goes to infinity. That is, we send information down the channel at a rate R smaller than χ , so that there are $N_n \simeq 2^{nR}$ possible randomly selected codewords \vec{j} that could be sent down over n uses. Each codeword gives rise to $\sim 2^{n \sum_j p_j S(\rho_j)}$ possible typical outputs $|\vec{k}\rangle_{\vec{j}}$. As always with Shannon-like random coding arguments [27], our set of possible outputs only occupy a fraction $2^{-n(\chi - R)}$ of the full output space. This sparseness of the actual outputs in the full space is the key to obtaining asymptotic zero error probability: all our error probabilities will scale as $2^{-n(\chi - R)}$.

The codeword sent down the channel is some typical sequence \vec{j} , which yields some typical output $|\vec{k}\rangle_{\vec{j}}$ with probability $p_{\vec{k}|\vec{j}}$. We begin with a von Neumann measurement corresponding to projectors $P, \mathbb{1} - P$ to check whether the output lies in the typical subspace \mathcal{H} . From Eq. (3) we can conclude that for any $\epsilon > 0$, for sufficiently large n , this measurement yields the result “yes” with probability larger than $1 - \epsilon$. We follow this with a binary projective measurement with projectors

$$P_{\vec{k}_1|\vec{j}_1} \equiv |\vec{k}_1\rangle_{\vec{j}_1} \langle \vec{k}_1|, \quad \mathbb{1} - P_{\vec{k}_1|\vec{j}_1}, \quad (7)$$

to check whether the input was \vec{j}_1 and the output was \vec{k}_1 . If this measurement yields the result “yes”, we conclude that the input was indeed \vec{j}_1 . Usually, however, this measurement yields the result “no”. In this case, we perform another measurement to check for typicality, and move on to a second trial output state, e.g., $|\vec{k}_2\rangle_{\vec{j}_1}$. If this measurement yields the result “yes”, we conclude that the input was \vec{j}_1 . Usually, of course, the measurement yields the result “no”, and so we project again and move on to a third trial output state, $|\vec{k}_3\rangle_{\vec{j}_1}$ etc. Having exhausted the $O(2^{n \sum_k p_k S(\rho_k)})$ typical output states from the codeword \vec{j}_1 , we turn to the typical output states from the input \vec{j}_2 , then \vec{j}_3 , and so on, moving through the $N_n \simeq 2^{nR}$ codewords until we eventually find a match. The maximum number of measurements that must be performed is hence

$$M \simeq 2^{nR} 2^{n \sum_k p_k S(\rho_k)}. \quad (8)$$

The probability amplitude that after m trials without finding the correct state, we find it at the $m + 1$ ’th trial can then be expressed as

$$\mathcal{A}_m(\text{yes}) = \vec{j} \langle \vec{k} | P(\mathbb{1} - P_{\ell_m}) P \dots P(\mathbb{1} - P_{\ell_1}) P | \vec{k} \rangle_{\vec{j}}, \quad (9)$$

where for $q = 1, \dots, m$ the operators P_{ℓ_q} represent the first m elements $P_{\vec{k}_r|\vec{j}_s}$ that compose the decoding sequence of projectors. The error probability $P_{\text{err}}(\vec{j}, \vec{k})$ of

mistaking the vector $|\vec{k}\rangle_{\vec{j}}$ can then be bounded by considering the worst case scenario in which the codeword sent is the last one tested in the sequence. Since this is the worst that can happen, $|\mathcal{A}_M(yes)|$ with $m = M$, is the smallest possible, so that $P_{err}(\vec{j}, \vec{k}) \leq 1 - |\mathcal{A}_M(yes)|^2$. Recall that the input codewords \vec{j} are randomly selected from the set of typical input sequences, and \vec{k} 's are typical output sequences. Then, the average error probability for a randomly selected set of input codewords can be bounded as $\langle P_{err} \rangle \leq 1 - \langle |\mathcal{A}_M(yes)|^2 \rangle \leq 1 - |\langle \mathcal{A}_M(yes) \rangle|^2$. Here $\langle \dots \rangle$ represents the average over all possible codewords of a given selected codebook \mathcal{C}_n and the averaging over all possible codebooks of codewords. The Cauchy-Swarz inequality $\langle |\mathcal{A}_M(yes)|^2 \rangle \geq |\langle \mathcal{A}_M(yes) \rangle|^2$ was employed. The last term can be evaluated as

$$\begin{aligned} \langle \mathcal{A}_m(yes) \rangle &= \text{Tr} \left[P \left(\mathbb{1} - \sum_{\ell_m} \pi_{\ell_m} P_{\ell_m} \right) P \dots \right. \\ &\quad \left. P \left(\mathbb{1} - \sum_{\ell_1} \pi_{\ell_1} P_{\ell_1} \right) P \tilde{\rho} \right] = \text{Tr} \left[(P - \tilde{\rho})^m \tilde{\rho} \right] \\ &= \sum_{k=0}^m \binom{m}{k} (-1)^k \text{Tr} \left[\tilde{\rho}^{k+1} \right], \end{aligned} \quad (10)$$

where π_ℓ stands for the probability $p_{\vec{j}} p_{\vec{k}|\vec{j}}$, and where we used (6) and (7) to write $\tilde{\rho} = \sum_{\ell} \pi_{\ell} P P_{\ell} P$. To prove the optimality of our decoding, it is hence sufficient to show that $\langle \mathcal{A}_m(yes) \rangle \sim 1$ even when the number m of measurements is equal to its maximum possible value M of Eq. (8). Consider then Eqs. (4) and (5) which imply the inequalities

$$\text{Tr} \tilde{\rho}^j \leq \sum_{i=0}^{\dim(\mathcal{H})} \lambda_i^j \leq 2^{n[S(1-j) + \delta(1+j)]}. \quad (11)$$

Use this and Eq. (3) to rewrite Eq. (10) as

$$\begin{aligned} \langle \mathcal{A}_m(yes) \rangle &\geq \text{Tr} \tilde{\rho} + \sum_{k=1}^m \binom{m}{k} (-1)^k \text{Tr} \left[\tilde{\rho}^{k+1} \right] \\ &\geq 1 - \epsilon - \sum_{k=1}^m \binom{m}{k} 2^{n[-kS(\rho) + \delta(k+2)]} = 1 - \epsilon - \gamma, \end{aligned} \quad (12)$$

where $\gamma \equiv 2^{2n\delta}[(1 + \zeta_n)^m - 1]$, with $\zeta_n = 2^{n[-S(\rho) + \delta]}$. If $S(\rho) > \delta$, for large n we can write

$$(1 + \zeta_n)^m - 1 \simeq e^{m\zeta_n} - 1 \simeq m\zeta_n. \quad (13)$$

Hence, γ is asymptotically negligible as long as $2^{2n\delta} m \zeta_n$ is vanishing for $n \rightarrow \infty$. This yields the constraint

$$m \leq 2^{n(S(\rho) - \delta)} \quad \text{for all } m. \quad (14)$$

In particular, it must hold for M , the largest value of m given in (8). Imposing this, the decoding procedure yields a vanishing error probability if the rate R satisfies

$$R < \chi - \delta, \quad (15)$$

as required by the Holevo bound [5].

Summarizing, we have shown that under the condition (15) the average amplitude $\langle \mathcal{A}_m(yes) \rangle$ of identifying the correct codeword is asymptotically close to 1 even in the worst case in which we had to check over *all* the other codewords $m = M$. This implies that the average probability of error in identifying the codeword asymptotically vanishes. In other words, the procedure works even when the measurements are chosen so that the codeword sent is the last one tested in the sequence of tests. Note that the same results presented here can be obtained also starting from the direct calculation of the error probability [23] (instead of using the probability amplitude).

We conclude by noting that from Eq. (9) one immediately sees that the POVM $\{E_\ell\}$ relative to the global decoding procedure is

$$\begin{aligned} E_1 &= P P_1 P; \quad E_2 = P(\mathbb{1} - P_1) P P_2 P(\mathbb{1} - P_1) P; \\ E_\ell &= P(\mathbb{1} - P_1) P(\mathbb{1} - P_2) P \dots P(\mathbb{1} - P_{\ell-1}) P \\ &\quad \times P_\ell \dots (\mathbb{1} - P_1) P; \quad E_0 = \mathbb{1} - \sum_{\ell=1}^M E_\ell, \end{aligned} \quad (16)$$

where P_ℓ is defined as in (7) and E_0 is the ‘‘abort’’ result. We gave a simple realization of this POVM using sequential ‘‘yes/no’’ projections, but different realizations may be possible. It is an alternative to the conventional Pretty-Good-Measurement. Note also that, with the exception of E_0 , all the operators in this POVM are simply projections onto pure states or on their orthogonal complement. Such sequence of projective measurements is ‘‘asymptotically unentangling’’ in the sense that the output state departs at most infinitesimally from its original separable form throughout the entire decoding procedure. This clarifies that the role of entanglement in the decoding is analogous to [29]: namely, increasing the distinguishability of a multi-partite set of states that are not orthogonal when considered by separate parties that do not employ entanglement.

Conclusions:— Using projective measurements in a sequential fashion, we gave a new proof that it is possible to attain the Holevo capacity when a noisy quantum channel is used to transmit classical information. Such measurements provide an alternative to the usual Pretty-Good-Measurements for channel decoding, and can be used in many of the same situations. In particular, an analogous procedure can be used to decode channels that transmit quantum information, to approach the coherent information limit [30–32]. This follows simply from the observation [32] that the transfer of quantum messages over the channel can be formally treated as a transfer of classical messages imposing an extra constraint of privacy in the signaling.

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